The quantum theory of a quadratic gravity action for heterotic strings

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Abstract. The wave function for the quadratic gravity theory derived from the heterotic string effective action is deduced to first order in $\frac{e^{-\Phi}}{g_4^2}$ by solving a perturbed second-order Wheeler-DeWitt equation, assuming that the potential is slowly varying with respect to Φ. Predictions for inflation based on the solution to the second-order Wheeler-DeWitt equation continue to hold for this higher-order theory. It is shown how formal expressions for the average paths in minisuperspace $\{\langle a(t)\rangle, \langle\Phi(t)\rangle\}$ for this theory can be used to determine the shifts from the classical solutions $a_{cl}(t)$ and $\Phi_{cl}(t)$, which occur only at third order in the expansion of the functional integrals representing the expectation values.

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Given a dimensionless metric and scalar field, the form of a sigma model coupled to gravity [1], renormalizable in the generalized sense, is

$$\begin{aligned} I \ = \ \int d^4x \ \sqrt{-g} \ \Big[& b_1(\phi)(\Box\phi)^2 + b_2(\phi)(\nabla_\mu\phi)(\nabla^\mu\phi)\Box\phi + b_3(\phi)[(\nabla_\mu\phi)(\nabla^\mu\phi)]^2 \\ & + \ b_4(\phi)(\nabla_\mu\phi)(\nabla^\mu\phi) \ + \ b_5(\phi) \ + \ c_1(\phi)R(\nabla_\nu\phi)(\nabla^\mu\phi) \\ & + \ c_2(\phi)R^{\mu\nu}(\nabla_\mu\phi)(\nabla_\nu\phi) \ + \ c_3(\phi)R\Box\phi + a_1(\phi)R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} \\ & + a_2(\phi)R_{\mu\nu}R^{\mu\nu} \ + \ a_3(\phi)R^2 \ + \ a_4(\phi)R \Big] \ + \ surface \ terms \end{aligned} \tag{1}$$

where $[b_4(\phi)] = 2$, $[b_5(\phi)] = 4$ and $[a_4(\phi)] = 2$, with $a_i(\phi) \neq 0$ for at least one $i \in \{1, 2, 3\}$ and a renormalizable potential term $b_5(\phi)$. Setting

$$\begin{aligned} & b_1(\phi) = b_2(\phi) = b_3(\phi) = 0 \\ & b_4(\phi) \ = \ \frac{1}{2\kappa^2} \qquad b_5(\phi) = -\frac{1}{\kappa^2}\tilde{V}(\phi) \\ & c_1(\phi) \ = \ c_2(\phi) \ = \ c_3(\phi) = 0 \\ & a_1(\phi) \ = \ \frac{e^{\frac{-\phi}{\kappa}}}{4g_4^2} \quad a_2(\phi) \ = \ -\frac{e^{\frac{-\phi}{\kappa}}}{g_4^2} \quad a_3(\phi) \ = \ \frac{e^{\frac{-\phi}{\kappa}}}{4g_4^2} \quad a_4(\phi) \ = \ \frac{1}{\kappa^2} \end{aligned} \tag{2}$$

produces an action describing an exponential coupling of a scalar field to quadratic curvature terms. Defining the dilaton field to have dimension 1, $\Phi = \frac{\phi}{\kappa}$, and $V(\Phi) \ = \ \frac{1}{\kappa^2}\tilde{V}(\phi)$, the quadratic gravity theory

$$I = \int d^4x \ \sqrt{-g} \left[\frac{1}{\kappa^2}R + \frac{1}{2}(D\Phi)^2 + \frac{e^{-\Phi}}{4g_4^2} \times (R_{\mu\nu\kappa\lambda}R^{\mu\nu\kappa\lambda} - 4R_{\mu\nu}R^{\mu\nu} + R^2) - V(\Phi) \right] \tag{3}$$

is equivalent to the one-loop heterotic string effective action with the coefficient of the $R\tilde{R}$ term set to zero. A non-minimal coupling $\xi R\phi^2$ term, which has been found useful for generating open inflationary universe models [2], can be added to this action by setting the coefficient $a_4(\phi)$ equal to $\frac{1}{\kappa^2}(1 + \xi\phi^2)$.

A conformal transformation of the R^2 term produces a Ricci scalar plus an extra scalar field, and any factor multiplying R can be eliminated through another conformal transformation [3-5]. There is also a Legendre transformation of the $R^{\mu\nu}R_{\mu\nu}$ term to a Ricci scalar together with extra tensor modes [6-8]. The scalar and tensor modes contain an intricate dependence on the derivatives of the metric and Ricci tensors, which complicates the definition of the corresponding momenta, so that there is no simplification in the canonical quantization procedure by applying a conformal or Legendre transformation to the action. The $R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma}$ can be rewritten as a linear combination of $C^{\mu\nu\rho\sigma}C_{\mu\nu\rho\sigma}$ and terms containing $R_{\mu\nu}$ and R. The renormalizability of the action with $C^{\mu\nu\rho\sigma}C_{\mu\nu\rho\sigma}$ has been established [9]. Its effect on unitarity must be counterbalanced by that of the other tensor modes.

For an action with a polynomial coupling between the scalar field and the quadratic curvature term $a_1(\phi)$, $a_2(\phi)$ and $a_3(\phi)$ will be shifted to $\tilde{a}_1(\phi)$, $\tilde{a}_2(\phi)$ and $\tilde{a}_3(\phi)$. The coefficients in a truncated Taylor series representation of $\frac{e^{-\Phi}}{g_4^2}$, $\sum_{n=0}^{N_0} \ \frac{1}{n!}\frac{(-1)^n}{g_4^{2n}}\Phi^n + \frac{1}{(N_0+1)!}(-1)^{N_0+1}\frac{e^{-\lambda\Phi}}{g_4^2}$,

$\lambda < 1$, would be shifted so that the couplings of the renormalized theory are $\sum_{n=0}^{N_0'} \frac{1}{n!} c_{i,n} \Phi^n + \frac{1}{(N_0!+1)!} \tilde{a}_i^{(N_0'+1)}(\lambda' \Phi),\ \lambda' < 1$. The remainder term can be made arbitrarily small and the series converges as $N_0' \to \infty$ unless $c_{i,n}$ has a factorial dependence on n, so that the new action with coefficients $\tilde{a}_i(\Phi),\ i = 1, 2, 3$, will be defined and renormalizable in the generalized sense.

After imposing the restriction to the minisuperspace of Friedmann-Robertson-Walker metrics, and adding a boundary term to the action to eliminate terms containing $\ddot{a}$, the one-dimensional action is

$$
\begin{aligned}
I \ = \ \int \, dt \Big[& 6a(-\dot{a}^2 + K) \ + \ \frac{1}{2} a^3 \dot{\Phi}^2 \\
& + \ \frac{e^{-\Phi}}{g_4^2} \dot{\Phi} \dot{a} (\dot{a}^2 + 3K) \ - \ a^3 V(\Phi) \Big].
\end{aligned} \tag{4}
$$

where $a(t)$ is the scale factor of the Friedmann-Robertson-Walker universe, which is open, flat or closed if $K = -1, 0$ or 1.

Given the conjugate momenta

$$
\begin{aligned}
P_a \ &= \ -12 a \dot{a} + 6 \frac{e^{-\Phi}}{g_4^2} \dot{\Phi} (\dot{a}^2 + K) \\
P_\Phi \ &= \ a^3 \dot{\Phi} \ + \ 2 \frac{e^{-\Phi}}{g_4^2} \dot{a} (\dot{a}^2 + 3K)
\end{aligned} \tag{5}
$$

the Wheeler-De Witt equation to first order in $\frac{e^{-\Phi}}{g_4^2}$ [10] is

$$
\begin{aligned}
H\Psi \ &= \ (H_0 + \frac{e^{-\Phi}}{g_4^2} H_1)\Psi \ = \ 0 \\
H_0 \ &= \ \frac{1}{24} \frac{\partial}{\partial a} \frac{1}{a} \frac{\partial}{\partial a} \ - \ \frac{1}{2a^3} \frac{\partial^2}{\partial \Phi^2} \ - \ 6aK \ + \ a^3 V(\Phi) \\
H_1 \ &= \ \frac{1}{a^4} \left(\frac{K}{4} - \frac{1}{576 a^4} \right) \frac{\partial}{\partial a} \ + \ \frac{1}{576 a^7} \frac{\partial^2}{\partial a^2} \ - \ \frac{1}{1728 a^6} \frac{\partial^3}{\partial a^3} \\
& \quad + \ \frac{1}{a^5} \left(\frac{7K}{4} - \frac{35}{576 a^4} \right) \frac{\partial}{\partial \Phi} \ + \ \frac{1}{24 a^8} \frac{\partial^2}{\partial a \partial \Phi} \ - \ \frac{1}{64 a^7} \frac{\partial^3}{\partial a^2 \partial \Phi} \ + \ \frac{1}{864 a^6} \frac{\partial^4}{\partial a \partial \Phi}.
\end{aligned} \tag{6}
$$

Given that $\Psi \doteq \Psi_0 + \frac{e^{-\Phi}}{g_4^2} \Psi_1$, when $|V^{-1} V'(\Phi)| \ll 1$ and $|V(\Phi)| \ll 1$ in Planck units, so that the derivatives of the quantum cosmological wave function Ψ with respect to Φ are negligible, the equation becomes $H_0 \Psi_1 \approx -H_1 \Psi_0$, to first order in $\frac{e^{-\Phi}}{g_4^2}$. It may be noted that $H_0 \left(\frac{e^{-\Phi}}{g_4^2} \Psi_1 \right)$ would contain a term of the form $\frac{1}{2a^3} \frac{\partial^2}{\partial \Phi^2} \left(\frac{e^{-\Phi}}{g_4^2} \Psi_1 \right)$ giving rise to a contribution $\frac{e^{-\Phi}}{g_4^2} \frac{1}{2a^3} \Psi_1 - \frac{1}{a^3} \frac{e^{-\Phi}}{g_4^2} \frac{\partial \Psi_1}{\partial \Phi} + \frac{1}{2a^3} \frac{e^{-\Phi}}{g_4^2} \frac{\partial^2 \Psi_1}{\partial \Phi^2}$ to the differential equation at first order in the expansion parameter. However, if this term is included, then the solution to the standard Wheeler-DeWitt equation with the $\frac{\partial^2}{\partial \Phi^2}$ operator and each of the Φ derivative terms in H_1 would have to be used. When the Φ derivative terms are discarded from at the beginning of the computation, it is sufficient to consider the differential equation for Ψ_1 without the additional term.

The correction to the wave function is then given by $\frac{e^{-\Phi}}{g_4^2}\Psi_1$ where

$$\Psi_1 \approx C_1\Psi_{01} + C_2\Psi_{02} - \Psi_{02}\int \Psi_{01}\frac{H_1\Psi_0}{W}a\ da + 24\Psi_{01}\int \Psi_{02}\frac{H_1\Psi_0}{W}a\ da \tag{7}$$

with the Wronskian defined to be

$$W = \Psi_{01}\frac{d}{da}\Psi_{02} - \Psi_{02}\frac{d}{da}\Psi_{01}. \tag{8}$$

Consistency with symmetries of the theory depends on the choice of boundary condition, which also determines the feasibililty of obtaining an inflationary cosmology. Upon consideration of the N=1 supergravity theory restricted to the minisuperspace of Bianchi IX metrics, for example, the requirement of homogeneity implies a Lie derivative condition on the spinor fields, which defines a no-boundary ground state [11]. While the no boundary wave function is defined to be regular in the limit $a \to 0$, this property only holds for the tunneling wave function when the operator ordering parameter p is less than one.

The no-boundary wave function is

$$\Psi_{0NB} = \frac{Ai\left(K\left(\frac{36}{V}\right)^{\frac{2}{3}}\left(1-\frac{a^2V}{6K}\right)\right)}{Ai\left(K\left(\frac{36}{V}\right)^{\frac{2}{3}}\right)} \tag{9}$$

when $K = -1$ or 1. While the no-boundary wave function is defined by a path integral over compact four-manifolds, leading to the conventional choice $K = 1$, the other values of K are possible if the range of coordinates in the flat or hyperbolic sections is finite. While the probability amplitude defined by the no-boundary wave function with a positive coefficient in the exponential prefactor does not directly imply the existence of an inflationary universe with the appropriate e-folding factor, it may be noted that a negative coefficient can be obtained by the other choice of sign of $(-z_0)^{\frac{3}{2}}$ in the asymptotic expansion of $Ai(-z_0)$, $z_0 = z(a=0) = -K\left(\frac{36}{V}\right)^{\frac{2}{3}}$ as $V(\Phi) \to 0$, so that as $z_0 \to -\infty$, the normalization factor tends to $\frac{1}{2\sqrt{\pi}}(-z_0)^{-\frac{1}{4}}e^{\mp\frac{2}{3}(-z_0)^{\frac{3}{2}}}$ if $K = 1$. Even if the coefficient is positive initially, the value of $V(\Phi)$ would be driven to zero, so that the change in sign can be obtained by analytic continuation in the variable V. The change in sign of the exponential prefactor does not affect the regularity of the wave function in the $a(t) \to 0$ limit. When $K = -1$, $Ai(-z_0) \to \frac{1}{\sqrt{\pi}}(z_0)^{-\frac{1}{4}}sin\ \left[\frac{2}{3}z_0^{\frac{3}{2}} + \frac{\pi}{4}\right]$, so that the wave function diverges $\frac{24}{V} + \frac{\pi}{4}$ tends to $n\pi$, n integer.

When $K = 0$, it is not necessary to rescale $a(t)$ and $V(\Phi)$ to obtain the standard form for the second-order Wheeler-DeWitt equation. Without a term proportional to K, one definition of z could be $(2V)^{-\frac{2}{3}}(a^2V) = 2^{-\frac{2}{3}}a^2V^{\frac{1}{3}}$, but then $z(a=0)$ would vanish. If the normalization factor is chosen to be $Ai(-z_c)$, where $z_c = z(a_c)$, it can be shown that the probability distribution is peaked at $V = 0$ if $a_c \ll 1$. The change in sign of z across the $V(\Phi) = 0$ boundary leads to different asymptotics for the wave function. If the positive sign is chosen for the exponent in the prefactor $e^{\mp\frac{2}{3}(-z_c)^{\frac{3}{2}}} = e^{\mp\frac{1}{3}a_c^3|V|^{\frac{1}{2}}}$, inflation again can be obtained.

Given the two independent solutions of the homogeneous second-order differential equation $H_0\Psi_0 = 0$

$$\begin{aligned} \Psi_{01} &= Ai(-z) \qquad \Psi_{02} = Bi(-z) \\ z &= -K\left(\frac{36}{V}\right)^{\frac{2}{3}}\left(1-\frac{a^2V}{6K}\right) \end{aligned} \tag{10}$$

the Wronskian [14] is

$$\begin{aligned} W &= Ai(-z)\frac{d}{da}Bi(-z) - Bi(-z)\frac{d}{da}Ai(-z) = \frac{dz}{da}\left[Ai(-z)\frac{d}{dz}Bi(-z) - Bi(-z))\frac{d}{dz}Ai(-z)\right] \\ &= -\frac{dz}{da}\pi^{-1} = \frac{aV^{\frac{1}{3}}}{3\pi}(36)^{\frac{2}{3}} \end{aligned} \tag{11}$$

and

$$\begin{aligned} \Psi_1 &= C_1Ai(-z) + C_2Bi(-z) \\ &- \frac{72\pi}{Ai\left(K\left(\frac{36}{V}\right)^{\frac{2}{3}}\right)}Bi(-z)\cdot\int\frac{da}{a^3}\left[KAi'(-z) - \frac{1}{36}\left(\frac{V}{36}\right)^{\frac{2}{3}}Ai'''(-z)\right]Ai(-z) \\ &+ \frac{72\pi}{Ai\left(K\left(\frac{36}{V}\right)^{\frac{2}{3}}\right)}Ai(-z)\cdot\int\frac{da}{a^3}\left[KAi'(-z) + \frac{1}{36}\left(\frac{V}{36}\right)^{\frac{2}{3}}Ai'''(-z)\right]Bi(-z). \end{aligned} \tag{12}$$

From Airy's differential equation, it follows that $A'''(-z)$ can be replaced by $zAi'(-z)+Ai(-z)$ in the integral, giving

$$\begin{aligned} \Psi_1 &= C_1Ai(-z) + C_2Bi(-z) \\ &- \frac{72\pi Bi(-z)}{Ai\left(K\left(\frac{36}{V}\right)^{\frac{2}{3}}\right)}\cdot\int\frac{da}{a^3}\left\{\left[\left(-K+\frac{1}{36}\left(\frac{V}{36}\right)^{\frac{2}{3}}z\right)Ai'(-z)+\frac{1}{36}\left(\frac{V}{36}\right)^{\frac{2}{3}}Ai(-z)\right]Ai(-z)\right\} \\ &+ \frac{72\pi Ai(-z)}{Ai\left(K\left(\frac{36}{V}\right)^{\frac{2}{3}}\right)}\cdot\int\frac{da}{a^3}\left\{\left[\left(-K+\frac{1}{36}\left(\frac{V}{36}\right)^{\frac{2}{3}}z\right)Ai'(-z)+\frac{1}{36}\left(\frac{V}{36}\right)^{\frac{2}{3}}Ai(-z)\right]Bi(-z)\right\}. \end{aligned} \tag{13}$$

Imposing the Hartle-Hawking boundary condition on the corrected wave function implies that it must have the same form as the standard wave function, so that the coefficient of $Bi(-z)$ should vanish.

After changing the integration variable, $\frac{da}{a^3} = \frac{3V^{\frac{5}{3}}}{(36)^{\frac{2}{3}}K^2}\left(1+\left(\frac{V}{36}\right)^{\frac{2}{3}}\frac{z}{K}\right)^{-2}dz$, the wave function Ψ_1 can be obtained by evaluating two integrals of the form $\int \frac{dz}{(1+kz)^2}Ai'(-z)Ai(-z)(1+k'z)$ and $\int \frac{dz}{(1+kz)^2}[Ai(-z)]^2$.

The integral $\int\ dz\ \sigma(z)\ S_\mu(\phi(z))\cdot S_\nu(\psi(z))$ has the form $[A(z)S_\mu(\phi(z))+BS_{\mu+1}(\phi(z))]\ S_\nu(\psi(z)) +[C(z)S_\mu(\phi(z))+D(z)S_{\mu+1}(\phi(z))]S_{\nu+1}(\psi(z))$ when $S_\mu(z)$ is a cylinder function [15] if

$$\begin{aligned}
\sigma(z) &= A'(z)\ +\ \left(\mu\frac{\phi'(z)}{\phi(z)}+\nu\frac{\psi'(z)}{\psi(z)}\right)A(z)+B\phi'(z)+C\psi'(z) \\
0 &= B'(z)\ +\ \left[\nu\frac{\psi'(z)}{\psi(z)}-(\mu+1)\frac{\phi'(z)}{\phi(z)}\right]B(z)\ +\ D\psi'(z)\ -\ A\phi'(z) \\
0 &= C'(z)\ +\ \left[\mu\frac{\phi'(z)}{\phi(z)}-(\nu+1)\frac{\psi'(z)}{\psi(z)}\right]C(z)\ +\ D\phi'(z)\ -\ A\psi'(z) \\
0 &= D'(z)\ -\ \left[(\mu+1)\frac{\phi'(z)}{\phi(z)}+(\nu+1)\frac{\psi'(z)}{\psi(z)}\right]D(z)\ -\ B\psi'(z)\ -\ C\phi'(z).
\end{aligned} \tag{14}$$

This set of coupled differential equations can be reduced to the 3×3 system

$$\frac{d}{dz}\begin{pmatrix} A(z) \\ B(z) \\ C(z) \end{pmatrix}\ +\ M\begin{pmatrix} A(z) \\ B(z) \\ C(z) \end{pmatrix}\ =\ \begin{pmatrix} a(z) \\ b(z) \\ c(z) \end{pmatrix} \tag{15}$$

where

$$\begin{aligned}
M &= \begin{pmatrix} (\mu+\nu)\frac{\phi'(z)}{\phi(z)} & \phi'(z) & \phi'(z) \\ -\phi'(z) & -(1+2\mu)\frac{\phi'(z)}{\phi(z)}+\frac{1}{\mu-\nu}\phi(z)\phi'(z) & (\mu+\nu)\frac{\phi'(z)}{\phi(z)}-\frac{1}{\mu-\nu}\phi(z)\phi'(z) \\ -\phi'(z) & \frac{1}{\mu-\nu}\phi(z)\phi'(z) & (\mu-\nu-1)\frac{\phi'(z)}{\phi(z)}-\frac{1}{\mu-\nu}\phi(z)\phi'(z) \end{pmatrix} \\
\begin{pmatrix} a(z) \\ b(z) \\ c(z) \end{pmatrix} &= \begin{pmatrix} \sigma(z) \\ (\mu+\nu)\phi'(z)-\frac{1}{\mu-\nu}\phi(z)^2\cdot\phi'(z) \\ -\frac{1}{\mu-\nu}\phi(z)^2\cdot\phi'(z) \end{pmatrix}.
\end{aligned} \tag{16}$$

The solution to this system of differential equations is

$$\begin{aligned}
\begin{pmatrix} A(z) \\ B(z) \\ C(z) \end{pmatrix} &= exp\left(-\int^z\ M(z')dz'\right)\cdot\begin{pmatrix} \int^z\ a(z')dz' \\ \int^z b(z')dz' \\ \int^z\ c(z')dz' \end{pmatrix} \\
&= T^{-1}Texp\left(-\int^z\ M(z')dz'\right)\cdot T^{-1}T\begin{pmatrix} \int^z\ a(z')dz' \\ \int^z\ b(z')dz' \\ \int^z\ c(z')dz' \end{pmatrix}.
\end{aligned} \tag{17}$$

If T is the matrix which diagonalizes $\int^z\ M(z')dz'$, then

$$Te^{\left(-\int^z\ M(z')dz'\right)}T^{-1}\ =\ exp\left(-T\int^z\ M(z'dz')T^{-1}\right)\ =\ \begin{pmatrix} e^{-\lambda_1} & 0 & 0 \\ 0 & e^{-\lambda_2} & 0 \\ 0 & 0 & e^{-\lambda_3} \end{pmatrix} \tag{18}$$

for some set of eigenvalues $\lambda_1,\lambda_2,\ \lambda_3$, and

$$\begin{pmatrix} A(z) \\ B(z) \\ C(z) \end{pmatrix}\ =\ T^{-1}\begin{pmatrix} e^{-\lambda_1} & 0 & 0 \\ 0 & e^{-\lambda_2} & 0 \\ 0 & 0 & e^{-\lambda_3} \end{pmatrix}T\begin{pmatrix} \int^z\ a(z')dz' \\ \int^z\ b(z')dz' \\ \int^z\ c(z')dz' \end{pmatrix}. \tag{19}$$

Since $Ai(z) = \frac{1}{3}\sqrt{z}\left[I_{-\frac{1}{3}}\left(\frac{2}{3}z^{\frac{3}{2}}\right) - I_{\frac{1}{3}}\left(\frac{2}{3}z^{\frac{3}{2}}\right)\right]$ and $Ai'(z) = -\frac{1}{3}z\left[I_{-\frac{2}{3}}\left(\frac{2}{3}z^{\frac{3}{2}}\right) - I_{\frac{2}{3}}\left(\frac{2}{3}z^{\frac{3}{2}}\right)\right]$

$$\begin{aligned}
\int dz\ \frac{(1+k'z)}{(1-kz)^2}Ai(z)Ai'(z) \\
= -\frac{1}{9}\int dz\ \frac{(1+c'z)}{(1-cz)^2}z^{\frac{3}{2}}\Bigg[I_{-\frac{1}{3}}\left(\frac{2}{3}z^{\frac{3}{2}}\right)I_{-\frac{2}{3}}\left(\frac{2}{3}z^{\frac{3}{2}}\right) - I_{\frac{1}{3}}\left(\frac{2}{3}z^{\frac{3}{2}}\right)I_{-\frac{2}{3}}\left(\frac{2}{3}z^{\frac{3}{2}}\right) \\
- I_{-\frac{1}{3}}\left(\frac{2}{3}z^{\frac{3}{2}}\right)I_{\frac{2}{3}}\left(\frac{2}{3}z^{\frac{3}{2}}\right) + I_{\frac{1}{3}}\left(\frac{2}{3}z^{\frac{3}{2}}\right)I_{\frac{2}{3}}\left(\frac{2}{3}z^{\frac{3}{2}}\right)\Bigg] \\
\int \frac{dz}{(1-kz)^2}[Ai(z)]^2 = \int \frac{dz}{(1-kz)^2}\ z\Bigg[I_{-\frac{1}{3}}\left(\frac{2}{3}z^{\frac{3}{2}}\right)I_{-\frac{1}{3}}\left(\frac{2}{3}z^{\frac{3}{2}}\right) - 2I_{-\frac{1}{3}}\left(\frac{2}{3}z^{\frac{3}{2}}\right)I_{\frac{1}{3}}\left(\frac{2}{3}z^{\frac{3}{2}}\right) \\
+ I_{\frac{1}{3}}\left(\frac{2}{3}z^{\frac{3}{2}}\right)I_{\frac{1}{3}}\left(\frac{2}{3}z^{\frac{3}{2}}\right)\Bigg]
\end{aligned} \tag{20}$$

with $\phi(z) = \frac{2}{3}z^{\frac{3}{2}}$.

Substituting this function of z into the matrix $M(z)$ and integrating gives

$$\int^z M(z')dz' = \begin{pmatrix} \frac{3}{2}(\mu+\nu)ln\ z & \frac{2}{3}z^{\frac{3}{2}} & \frac{2}{3}z^{\frac{3}{2}} \\ -\frac{2}{3}z^{\frac{3}{2}} & -\frac{3}{2}(1+2\mu)ln\ z + \frac{2}{9}\frac{1}{\mu-\nu}z^3 & \frac{3}{2}(\mu+\nu)ln\ z - \frac{2}{9}\frac{1}{\mu-\nu}z^3 \\ -\frac{2}{3}z^{\frac{3}{2}} & \frac{2}{9}\frac{1}{\mu-\nu}z^3 & \frac{3}{2}(\mu-\nu-1)ln\ z - \frac{2}{9}\frac{1}{\mu-\nu}z^3 \end{pmatrix}. \tag{21}$$

The eigenvalues are roots of the cubic equation

$$\begin{aligned}
\lambda^3\ +\ 3\ ln\ z\ \lambda^2\ +\ \left[\frac{2}{3}z^3 ln\ z - \frac{9}{4}(3\mu^2+\nu^2+\mu+\nu-1)ln^2 z\right]\lambda\ +\ \frac{4}{3}(\mu+\nu+1)z^3 ln\ z \\
-\ (\mu+\nu)z^3 ln^2 z\ +\ \frac{27}{8}(\mu+\nu)(1+2\mu)(\mu-\nu-1)ln^3 z\ =\ 0.
\end{aligned} \tag{22}$$

Defining the coefficients $\alpha,\ \beta,\ \gamma$ by using the standard form of the cubic equation $\lambda^3+\alpha\lambda^2+\beta\lambda+\gamma = 0$, it follows that

$$\begin{aligned}
p\ &=\ -\frac{9}{4}\left(3\mu^2+\nu^2+\mu+\nu+\frac{1}{3}\right)ln^2 z\ +\ \frac{2}{3}z^3 ln\ z \\
q\ &=\ \frac{9}{8}\left(6\mu^3-6\mu\nu^2+3\mu^2-\nu^2-6\mu\nu-\mu-\nu-\frac{2}{9}\right)\ ln^3 z\ +\ \frac{1}{3}(\mu+\nu+2)z^3 ln^2 z.
\end{aligned} \tag{23}$$

When $\mu = -\frac{1}{3}$ and $\nu = -\frac{2}{3}$,

$$Q\ =\ \left(\frac{p}{3}\right)^3 + \left(\frac{q}{2}\right)^2\ =\ \frac{8}{729}z^9 ln^3 z\ +\ \frac{2}{81}z^6 ln^4 z\ +\ \frac{1}{216}z^3 ln^5 z\ -\ \frac{1}{1728}ln^6 z \tag{23}$$

and $Q = 0$ when $z = 1$ or if $\frac{8}{81}w^3 + \frac{2}{9}w^2 + \frac{1}{24}w - \frac{1}{192} = 0$ with $w = \frac{z^3}{ln\ z}$. Since this cubic equation has a single positive root at $w \doteq 0.08516$, $Q > 0$ for all $z > 1$. If $w < -2.12470$, then $Q < 0$; if

$-2.12469 < w < -0.29559$, then $Q > 0$; and if $-0.29558 < w < 0$, then $Q < 0$. The eigenvalues are

$$
\begin{aligned}
\lambda_1 &= \left[-\frac{1}{6}z^3ln^2z + \left[\frac{8}{729}z^9ln^3z + \frac{2}{81}z^6ln^4z + \frac{1}{216}z^3ln^5z - \frac{1}{1728}ln^6\right]^{\frac{1}{2}}\right]^{\frac{1}{3}} \\
&+ \left[-\frac{1}{6}z^3ln^2z - \left[\frac{8}{729}z^9ln^3z + \frac{2}{81}z^6ln^4z + \frac{1}{216}z^3ln^5z - \frac{1}{1728}ln^6z\right]^{\frac{1}{2}}\right]^{\frac{1}{3}} \\
\lambda_2 &= -\frac{1}{2}\Bigg\{\left[-\frac{1}{6}z^3ln^2z + \left[\frac{8}{729}z^9ln^3z + \frac{2}{81}z^6ln^4z + \frac{1}{216}z^3ln^5z - \frac{1}{1728}\right]^{\frac{1}{2}}\right]^{\frac{1}{3}} \\
&+ \left[-\frac{1}{6}z^3ln^2z - \left[\frac{8}{729}z^9ln^3z + \frac{2}{81}z^6ln^4z + \frac{1}{216}z^3ln^5z - \frac{1}{1728}ln^6z\right]^{\frac{1}{2}}\right]^{\frac{1}{3}}\Bigg\} \\
&+ \frac{i\sqrt{3}}{2}\Bigg\{\left[-\frac{1}{6}z^3ln^2z + \left[\frac{8}{729}z^9ln^3z + \frac{2}{81}z^6ln^4z + \frac{1}{216}z^3ln^5z - \frac{1}{1728}ln^6\right]^{\frac{1}{2}}\right]^{\frac{1}{3}} \\
&- \left[-\frac{1}{6}z^3ln^2z - \left[\frac{8}{729}z^9ln^3z + \frac{2}{81}z^6ln^4z + \frac{1}{216}z^3ln^5z - \frac{1}{1728}ln^6z\right]^{\frac{1}{2}}\right]^{\frac{1}{3}}\Bigg\} \qquad (25) \\
\lambda_3 &= -\frac{1}{2}\Bigg\{\left[-\frac{1}{6}z^3ln^2z + \left[\frac{8}{729}z^9ln^3z + \frac{2}{81}z^6ln^4z + \frac{1}{216}z^3ln^5z - \frac{1}{1728}\right]^{\frac{1}{2}}\right]^{\frac{1}{3}} \\
&+ \left[-\frac{1}{6}z^3ln^2z - \left[\frac{8}{729}z^9ln^3z + \frac{2}{81}z^6ln^4z + \frac{1}{216}z^3ln^5z - \frac{1}{1728}ln^6z\right]\right]^{\frac{1}{3}}\Bigg\} \\
&- \frac{i\sqrt{3}}{2}\Bigg\{\left[-\frac{1}{6}z^3ln^2z + \left[\frac{8}{729}z^9ln^3z + \frac{2}{81}z^6ln^4z + \frac{1}{216}z^3ln^5z - \frac{1}{1728}ln^6\right]^{\frac{1}{2}}\right]^{\frac{1}{3}} \\
&- \left[-\frac{1}{6}z^3ln^2z - \left[\frac{8}{729}z^9ln^3z + \frac{2}{81}z^6ln^4z + \frac{1}{216}z^3ln^5z - \frac{1}{1728}ln^6z\right]\right]^{\frac{1}{3}}\Bigg\}.
\end{aligned}
$$

Similarly, the following expressions are obtained for p, q and Q for the other values of μ and ν

$$
\begin{aligned}
p &= \frac{2}{3}z^3ln\ z - \frac{7}{4}ln^2z & \mu = \frac{1}{3},\ \nu = -\frac{2}{3} \\
q &= \frac{5}{9}z^3ln^2z + \frac{3}{2}ln^3z & \\
Q &= \frac{8}{729}z^3ln^3z - \frac{1}{108}z^6ln^4z + \frac{139}{216}z^3ln^5z + \frac{629}{1728}ln^6z & \\
p &= \frac{2}{3}z^3ln\ z - \frac{13}{4}ln^2z & \mu = -\frac{1}{3},\ \nu = \frac{2}{3} \\
q &= \frac{3}{2}ln^3z + \frac{7}{9}z^3ln^2z & \\
Q &= \frac{8}{729}z^9ln^3z - \frac{1}{108}z^6ln^4z + \frac{295}{216}z^3ln^5z - \frac{1225}{1728}ln^6z & \\
p &= \frac{2}{3}z^3ln\ z - \frac{19}{4}ln^2z & \mu = \frac{1}{3},\ \nu = \frac{2}{3} \\
q &= z^3ln^2z - \frac{15}{4}ln^3z & (26) \\
Q &= \frac{8}{729}z^9ln^3z - \frac{5}{324}z^6ln^4z - \frac{11}{54}z^3ln^5z - \frac{49}{108}ln^6z. &
\end{aligned}
$$

The transformation matrix T is equal to

$$
\begin{aligned}
T &= \begin{pmatrix} v_{11} & v_{21} & v_{31} \\ v_{12} & v_{22} & v_{32} \\ v_{13} & v_{23} & v_{33} \end{pmatrix} \\
\begin{pmatrix} v_{i1} \\ v_{i2} \\ v_{i3} \end{pmatrix} &= \frac{1}{\mathcal{N}_i} \begin{pmatrix} -\frac{2}{3} z^{\frac{3}{2}} \frac{\left[\lambda_i + \frac{3}{2}(1+3\mu+\nu) ln\ z - \frac{4}{9}\frac{1}{\mu-\nu} z^3\right]}{\left[\frac{4}{9}z^3 + \left[\frac{2}{9}\frac{1}{\mu-\nu}z^3 - \frac{3}{2}(\mu+\nu) ln\ z\right]\left[\lambda_i - \frac{3}{2}(\mu+\nu) ln\ z\right]\right]} \\ 1 \\ -\frac{\frac{4}{9}z^3 + \left[\lambda_i + \frac{3}{2}(1+2\mu) ln\ z - \frac{2}{9}\frac{1}{\mu-\nu}z^3\right]\left[\lambda_i - \frac{3}{2}(\mu+\nu) ln\ z\right]}{\left[\frac{4}{9}z^3 + \left[\frac{2}{9}\frac{1}{\mu-\nu}z^3 - \frac{3}{2}(\mu+\nu) ln\ z\right]\left[\lambda_i - \frac{3}{2}(\mu+\nu) ln\ z\right]\right]} \end{pmatrix}
\end{aligned}
\tag{27}
$$

where $\mathcal{N}_i$ is a normalization factor, so that

$$
\begin{aligned}
A(z) &= (e^{-\lambda_1} v_{11}^* v_{11} + e^{-\lambda_2} v_{12}^* v_{12} + e^{-\lambda_3} v_{13}^* v_{13}) \int^z a(z')dz' \\
&+ (e^{-\lambda_1} v_{11}^* v_{21} + e^{-\lambda_2} v_{12}^* v_{22} + e^{-\lambda_3} v_{13}^* v_{23}) \int^z b(z')dz' \\
&+ (e^{-\lambda_1} v_{11}^* v_{31} + e^{-\lambda_2} v_{12}^* v_{32} + e^{-\lambda_3} v_{13}^* v_{31}) \int^z c(z')dz' \\
B(z) &= (e^{-\lambda_1} v_{21}^* v_{11} + e^{-\lambda_2} v_{22}^* v_{12} + e^{-\lambda_3} v_{23}^* v_{13}) \int^z a(z')dz' \\
&+ (e^{-\lambda_1} v_{21}^* v_{21} + e^{-\lambda_2} v_{22}^* v_{22} + e^{-\lambda_3} v_{23}^* v_{23}) \int^z b(z')dz' \\
&+ (e^{-\lambda_1} v_{21}^* v_{31} + e^{-\lambda_2} v_{22}^* v_{32} + e^{-\lambda_3} v_{23}^* v_{31}) \int^z c(z')dz' \\
C(z) &= (e^{-\lambda_1} v_{31}^* v_{11} + e^{-\lambda_2} v_{32}^* v_{12} + e^{-\lambda_3} v_{33}^* v_{13}) \int^z a(z')dz' \\
&+ (e^{-\lambda_1} v_{31}^* v_{21} + e^{-\lambda_2} v_{32}^* v_{22} + e^{-\lambda_3} v_{33}^* v_{23}) \int^z b(z')dz' \\
&+ (e^{-\lambda_1} v_{31}^* v_{31} + e^{-\lambda_2} v_{32}^* v_{32} + e^{-\lambda_3} v_{33}^* v_{31}) \int^z c(z')dz'.
\end{aligned}
\tag{28}
$$

For the first integral in equation (20)

$$
\begin{aligned}
\int^z a(z')dz' &= \frac{1}{k^{\frac{5}{2}}}\left[\sqrt{kz}\frac{3-2kz}{1-kz} - sinh^{-1}\left[\left(\frac{kz}{1-kz}\right)^{\frac{1}{2}}\right]\right] + \frac{2}{k^{\frac{5}{2}}} ln\ \left(\frac{(1-kz)^{\frac{1}{2}}}{1+\sqrt{kz}}\right) + \frac{2}{3}\frac{k'}{k^2} z^{\frac{3}{2}} \\
&+ \frac{k'}{k^{\frac{7}{2}}}\left(\sqrt{kz}\frac{5-4kz}{1-kz} - sinh^{-1}\left[\left(\frac{kz}{1-kz}\right)^{\frac{1}{2}}\right]\right) + 4\frac{k'}{k^{\frac{7}{2}}} ln\ \left(\frac{(1-kz)^{\frac{1}{2}}}{1+\sqrt{kz}}\right) \\
\int^z b(z')dz' &= \frac{2}{3}(\mu+\nu) z^{\frac{3}{2}} - \frac{8}{81}\frac{1}{\mu-\nu} z^{\frac{9}{2}} \\
\int^z c(z')dz' &= -\frac{8}{81}\frac{1}{\mu-\nu} z^{\frac{9}{2}}.
\end{aligned}
\tag{29}
$$

The equality of the indices $\mu,\ \nu$ in the second integral gives rise to a singularity in the matrix $\int^z\ M(z')dz'$, which can avoided by using the identity $I_{\nu-1}(z) - I_{\nu+1}(z) = \frac{2\nu}{z} I_\nu(z)$, to obtain an

integral containing the products of modified Bessel functions different indices. The correction to the standard wave function can be obtained through the substitution $z \to -z$ in the integrals (20).

The contribution of the additional integral in equation (12) to the wave function includes $Ai(-z)$ so that the functional dependence of the normalization factor $Ai(-z_0)+C_1\frac{e^{-\Phi}}{g_4^2}Ai(-z_0)+\frac{e^{-\Phi}}{g_4^2}f(z_0)$ is divisible by $Ai(-z_0)$. Given that the normalization factor contains $Ai\left(K\left(\frac{36}{V}\right)^{\frac{2}{3}}\right)$, the conclusions concerning the feasibility of predicting an inflationary universe through the quantum cosmological wave function are unaltered by the addition of the higher-order curvature terms.

In the Planck era, the higher-order curvature terms in the perturbative expansion of the string effective action have approximately the same magnitude as the Ricci scalar, and similarly it is inappropriate to use a truncated form of a series expansion in $\frac{e^{-\Phi}}{g_4^2}$ of the Wheeler-De Witt equation. Instead, a closed-form sixth order differential equation, which can be obtained by including the conjugate momentum to derivative of the scale factor $P_{\dot a}$, and then using the Ostrogradski method to define the Hamiltonian for the higher-derivative theory, can be used to define the quantum cosmological wave function in the initial era. Given the one-dimensional action

$$I \;=\; \int\; dt\;\left[(6a^2\ddot a + 6a\dot a^2 + 6aK) + \frac{1}{2}a^3\dot\Phi^2 \;+\; 6\frac{e^{-\Phi}}{g_4^2}\ddot a(\dot a^2 + K)\right] \tag{30}$$

and the conjugate momenta

$$\begin{aligned} P_a \;&=\; \frac{\partial L}{\partial \dot a} - \frac{d}{dt}\left(\frac{\partial L}{\partial \ddot a}\right) \;=\; 6\frac{e^{-\Phi}}{g_4^2}\dot\Phi(\dot a^2+K) \\ P_{\dot a} \;&=\; \frac{\partial L}{\partial \ddot a} \;=\; 6\left(a^2 + \frac{e^{-\Phi}}{g_4^2}(\dot a^2+K)\right) \\ P_\Phi \;&=\; \frac{\partial L}{\partial \dot\Phi} \;=\; a^3\dot\Phi \end{aligned} \tag{31}$$

the Hamiltonian is

$$\begin{aligned} H \;&=\; P_a\dot a \;+\; P_{\dot a}\ddot a \;+\; P_\Phi\dot\Phi \;-\; L \\ &=\; -6a(\dot a^2+K) \;+\; 6\frac{e^{-\Phi}}{g_4^2}\dot\Phi(\dot a^2+K)\dot a \;+\; \frac{1}{2}a^3\dot\Phi^2 \\ &=\; -g_4^2P_\Phi^{-1}e^\Phi a^4P_a \;+\; \frac{1}{2a^3}P_\Phi^2 \;+\; \left[\frac{g_4^2}{6}P_\Phi^{-1}e^\Phi P_a^2a^3P_a \;-\; KP_a^2\right]^{\frac{1}{2}} \end{aligned} \tag{32}$$

and the pseudo-differential equation $H\Psi \;=\; 0$ can be transformed into the partial differential equation *

$$\begin{aligned} -\frac{g_4^2}{6}e^{-\Phi}\left(a^3\frac{\partial^4\Psi}{\partial a^3\partial\Phi} + 6a^2\frac{\partial^3\Psi}{\partial a^2\partial\Phi}\right) \;+\; Ke^{-2\Phi}\left(\frac{\partial^4\Psi}{\partial a^2\partial\Phi^2} - \frac{\partial^3\Psi}{\partial a^2\partial\Phi}\right) \\ =\; a^4g_4^4\left[4a^3\frac{\partial\Psi}{\partial a} + a^4\frac{\partial^2\Psi}{\partial a^2}\right] \;+\; ag_4^2e^{-\Phi}\left(\frac{\partial^4\Psi}{\partial a^2\partial\Phi^2} \;+\; \frac{\partial^3\Psi}{\partial a\partial\Phi^2}\right) \\ +\;\frac{3}{2}g_4^2e^{-\Phi}\left(a\frac{\partial^2\Psi}{\partial a\partial\Phi} \;-\; \frac{\partial^3\Psi}{\partial\Phi^3}\right) \;+\; \frac{1}{4a^6}e^{-\Phi}\frac{\partial}{\partial\Phi}\left(e^{-\Phi}\frac{\partial^5\Psi}{\partial\Phi^5}\right). \end{aligned} \tag{33}$$

* This equation differs from the equation (11) in reference [14] by a derivative with respect to Φ. Imposing an additional constraint on the wave function, the derivative term can be eliminated through the addition of an extra term in the Lagrangian.

By including a potential term in the Lagrangian and discarding terms containing derivatives of Ψ and $V(\Phi)$ with respect to Φ, the following equation is obtained

$$\frac{a^2 g_4^2}{6}\frac{d^3\Psi}{da^3} + a g_4^2(1 + a^6 e^{\Phi})\frac{d^2\Psi}{da^2} + [g_4^2(1 + a^6 e^{\Phi}) - 2g_4^2 a^6 V(\Phi)]\frac{d\Psi}{da} - 3g_4^2 a^5 V(\Phi)\Psi = 0 \tag{34}$$

The corrected wave function in the inflationary epoch can be matched with the solution to a third-order partial differential equation in the initial era along a boundary which also must be determined by setting the derivatives up to second order in a to be equal.

While the wave function $\Psi(a, \Phi)$ has been used to establish whether the curvature-dependence defined by the potential favours inflation, to determine the most probable path in minisuperspace $\{a(t), \Phi(t)\}$, it is preferable to consider the partition function

$$Z \ = \ \int \ e^{-I[a(t),\Phi(t)]} \ d[a(t)]d[\Phi(t)] \tag{35}$$

which is extremized at the classical solutions

$$\frac{\delta Z}{\delta a}\bigg|_{a_{cl.}} \ = \ \frac{\delta Z}{\delta \Phi}\bigg|_{\Phi_{cl.}} \ = \ 0. \tag{36}$$

The expectation values $\langle a(t)\rangle$ and $\langle \Phi(t)\rangle$ based on the action I are

$$\begin{aligned}
\langle a(t)\rangle \ &= \ \frac{\int \ a(t)e^{-I[a(t),\Phi(t)]}d[a(t)]d[\Phi(t)]}{\int \ e^{-I[a(t),\Phi(t)]} \ d[a(t)]d[\Phi(t)]} \\
&\simeq \ \frac{\int \ a(t)e^{-\left[\frac{\delta^2 I}{\delta a(t)^2}(\delta a(t))^2 \ + \ 2\frac{\delta^2 I}{\delta a(t)\delta\Phi(t)}\delta a(t)\delta\Phi(t) \ + \ \frac{\delta^2 I}{(\delta\Phi(t))^2}(\delta\Phi(t))^2\right]} \ d[a(t)]d[\Phi(t)]}{\int \ e^{-\left[\frac{\delta^2 I}{\delta a(t)^2}(\delta a(t))^2 \ + \ 2\frac{\delta^2 I}{\partial a(t)\partial\Phi(t)}\delta a(t)\delta\Phi(t) \ + \ \frac{\delta^2 I}{(\delta\Phi(t))^2}(\delta\Phi(t))^2\right]} \ d[a(t)]d[\Phi(t)]} \\
\langle \Phi(t)\rangle \ &= \ \frac{\int \ \Phi(t) \ e^{-I[a(t),\Phi(t)]}d[a(t)]d[\Phi(t)]}{\int \ e^{-I[a(t),\Phi(t)]} \ d[a(t)]d[\Phi(t)]} \\
&\simeq \ \frac{\int \ \Phi(t) \ e^{-\left[\frac{\delta^2 I}{\delta a(t)^2}(\delta a(t))^2 \ + \ 2\frac{\delta^2 I}{\delta a(t)\delta\Phi(t)}\delta a(t)\delta\Phi(t) \ + \ \frac{\delta^2 I}{(\delta\Phi(t))^2}(\delta\Phi(t))^2\right]} \ d[a(t)]d[\Phi(t)]}{\int \ e^{-\left[\frac{\delta^2 I}{\delta a(t)^2}(\delta a(t))^2 \ + \ 2\frac{\delta^2 I}{\delta a(t)\delta\Phi(t)}\delta a(t)\delta\Phi(t) \ + \ \frac{\delta^2 I}{(\delta\Phi(t))^2}(\delta\Phi(t))^2\right]} \ d[a(t)]d[\Phi(t)]}
\end{aligned} \quad . \tag{37}$$

Since

$$\begin{aligned}
\frac{\delta^2 I}{\delta a^2} \ &= \ \int \ dt \ \left[\frac{\partial^2 L}{\partial a^2} \ - \ \frac{3}{2} \ \frac{d}{dt}\left(\frac{\partial^2 L}{\partial a \partial \dot{a}}\right) \ + \ \frac{1}{2}\frac{d^2}{dt^2}\left(\frac{\partial^2 L}{\partial \dot{a}^2}\right)\right] \\
\frac{\delta^2 I}{\delta \Phi^2} \ &= \ \int \ dt \ \left[\frac{\partial^2 L}{\partial \Phi^2} \ - \ \frac{3}{2} \ \frac{d}{dt}\left(\frac{\partial^2 L}{\partial \Phi \partial \dot{\Phi}}\right) \ + \ \frac{1}{2}\frac{d^2}{dt^2}\left(\frac{\partial^2 L}{\partial \dot{\Phi}^2}\right)\right]
\end{aligned} \tag{38}$$

and the equation of motion for $a(t)$ implies that $\frac{\partial L}{\partial a} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{a}}\right) = \sum_{n\geq 1} \ f_n(a, \dot{a}, \Phi, \dot{\Phi})(a(t) - a_{cl}(t))^n$,

$$\begin{aligned}
\frac{\partial}{\partial \Phi}\left(\frac{\partial L}{\partial a} \ - \ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{a}}\right)\right) \ &= \ \sum_{n\geq 1} \ \frac{\partial f_n(a, \dot{a}, \Phi, \dot{\Phi})}{\partial \Phi}(a(t) - a_{cl}(t))^n \\
\frac{\partial}{\partial \dot{\Phi}}\left(\frac{\partial L}{\partial a} \ - \ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{a}}\right)\right) \ &= \ \sum_{n\geq 1} \ \frac{\partial f_n(a, \dot{a}, \Phi, \dot{\Phi})}{\partial \dot{\Phi}}(a(t) - a_{cl}(t))^n
\end{aligned} \tag{39}$$

both vanish when $a(t) = a_{cl}(t)$. Similarly, $\frac{\partial}{\partial a}\left(\frac{\partial L}{\partial \Phi} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\Phi}}\right)\right) = \frac{\partial}{\partial \dot{a}}\left(\frac{\partial L}{\partial \Phi} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\Phi}}\right)\right) = 0$ when $\Phi(t) = \Phi_{cl}(t)$. From the equation (4),

$$\begin{aligned}
\frac{\partial^2 L}{\partial a^2} - \frac{3}{2}\frac{d}{dt}\left(\frac{\partial^2 L}{\partial a \partial \dot{a}}\right) + \frac{1}{2}\frac{d^2}{dt^2}\left(\frac{\partial^2 L}{\partial \dot{a}}\right)^2 &= 3a\dot{\Phi}^2 - 6aV(\Phi) + 12\ddot{a} + \frac{6}{g_4^2}\Big(-3e^{-\Phi}\dot{\Phi}\ddot{\Phi}\dot{a} \\
&\quad - 2e^{-\Phi}\dot{\Phi}^2\ddot{a} + 2e^{-\Phi}\ddot{\Phi}\ddot{a} + e^{-\Phi}\frac{d^3\Phi}{dt^3}\dot{a} + e^{-\Phi}\dot{\Phi}\frac{d^3 a}{dt^3} \\
&\quad + e^{-\Phi}\dot{\Phi}^3\dot{a}\Big) \\
\frac{\partial^2 L}{\partial \Phi^2} - \frac{3}{2}\frac{d}{dt}\left(\frac{\partial^2 L}{\partial \Phi \partial \dot{\Phi}}\right) + \frac{1}{2}\frac{d^2}{dt^2}\left(\frac{\partial^2 L}{\partial \dot{\Phi}^2}\right) &= -a^3 V''(\Phi) + 2\frac{e^{-\Phi}}{g_4^2}\dot{\Phi}\dot{a}(\dot{a}^2 + 3K) \\
&\quad + 3\frac{e^{-\Phi}}{g_4^2}\dot{\Phi}\dot{a}(\dot{a}^2 + 3K) - 9\frac{e^{-\Phi}}{g_4^2}\ddot{a}(\dot{a}^2 + K) \\
&\quad + \frac{9}{4}(2a\dot{a}^2 + 3a^2\ddot{a}).
\end{aligned} \tag{40}$$

and substituting the approximate solutions to the equations of motion for $a(t)$ and $\Phi(t)$ [10], based on the heterotic string potential, *

$$\begin{aligned}
a(t) &= a_0 e^{\lambda t} \\
\Phi(t) &\sim ln\left|\frac{27}{4C}\frac{h^2 e^{-3\sigma_0}}{b_0^3 g_4^4}\, cosh(\sqrt{C}(t - t_0))\right| \\
18\lambda^2 - \frac{3}{2}C + \frac{3}{16}g_4^2 e^{-3\sigma_0}k^2 e^D &= 0 \\
c + h &= ke^{-\frac{\sqrt{C}}{2}t}
\end{aligned} \tag{41}$$

gives

$$\begin{aligned}
\frac{\partial^2 L}{\partial a^2} - \frac{3}{2}\frac{d}{dt}\left(\frac{\partial^2 L}{\partial a \partial \dot{a}}\right) + \frac{1}{2}\frac{d^2}{dt^2}\left(\frac{\partial^2 L}{\partial \dot{a}^2}\right) &\underset{t\to\infty}{\longrightarrow} \left[3a_0(1 + 4\lambda^2) + \frac{g_4^2 e^{-3\sigma_0}}{16}k^2\right]e^{\lambda t} + \mathcal{O}(e^{(\lambda - \sqrt{C})t}) > 0 \\
\frac{\partial^2 L}{\partial \Phi^2} - \frac{3}{2}\frac{d}{dt}\left(\frac{\partial^2 L}{\partial \Phi \partial \dot{\Phi}}\right) + \frac{1}{2}\frac{d^2}{dt^2}\left(\frac{\partial^2 L}{\partial \dot{\Phi}^2}\right) &\underset{t\to\infty}{\longrightarrow} \frac{9}{4}a_0^3\lambda^2(2 + 3\lambda^2)e^{3\lambda t} + \mathcal{O}\left(e^{(3\lambda - \sqrt{C})t}\right) > 0.
\end{aligned} \tag{42}$$

To second order, the probability distributions about the classical paths in minisuperspace would be Gaussian, and average values $\langle a(t)\rangle$, $\langle \Phi(t)\rangle$ equal $a_{cl}(t)$, $\Phi_{cl}(t)$. A shift in the expectation values $\langle a(t)\rangle$, $\langle \Phi(t)\rangle$ only arises at third order in the expansion of the functional integrals in equation (36).

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* The opposite sign to the usual convention must be chosen to obtain a positive value for $V(\Phi = 0)$.

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